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$$(V_1) \begin{vmatrix} a_0 & a_0' + a_1 & a_1' + a_2 & a_2' + a_3 & a_3' + a_4 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ \beta_0 & 2\beta_0' + \beta_1 & \beta_0'' + 2\beta_1' + \beta_2 & \beta_1'' + 2\beta_2' + \beta_3 & \beta_2'' + 2\beta_3' \\ 0 & \beta_0 & \beta_0' + \beta_1 & \beta_1' + \beta_2 & \beta_2' + \beta_3 \\ 0 & 0 & \beta_0 & \beta_1 & \beta_2 \end{vmatrix} \equiv 0,$$

$$(V_2) \begin{vmatrix} a_0 & a_0' + a_1 & a_1' + a_2 & a_2' + a_3 & a_4' \\ 0 & a_0 & a_1 & a_2 & a_3 \\ \beta_0 & 2\beta_0' + \beta_1 & \beta_0'' + 2\beta_1' + \beta_2 & \beta_1'' + 2\beta_2' + \beta_3 & \beta_3'' \\ 0 & \beta_0 & \beta_0' + \beta_1 & \beta_1' + \beta_2 & \beta_3' \\ 0 & 0 & \beta_0 & \beta_1 & \beta_3 \end{vmatrix} \equiv 0.$$

Hence  $V_1$  and  $V_2$  furnish the required necessary condition.

NEW YORK CITY, September, 1903.

## LINEAR COVARIANTS OF THE BINARY QUADRATIC AND CUBIC.

By L. C. WALKER, Professor of Mathematics, Colorado School of Mines, Golden, Col.

The definition of *weight* is that every coefficient is of weight  $w$  measured by its suffix, and that every product of coefficients is of weight measured by the sum of the suffixes of its various factors.

A semi-covariant of the two quantities is a function of the two sets of coefficients, which is homogeneous in each set separately, and *isobaric* (equal weight) on the whole, though not necessarily in the sets separately.

The practice of speaking of a covariant whose dimensions are partial degrees  $i_1, i_2$  in the two sets of coefficients and  $w$  in the variables has of late become almost universal.\*

The degrees of quantities in the variables are generally† spoken of as their orders  $p_1, p_2$ .

The order  $w$ , the partial degrees  $i_1, i_2$  in the coefficients of the binary quadratic and cubic

$$(a_0, a_1, a_2)(x, y)^2, \quad (b_0, b_1, b_2, b_3)(x, y)^3,$$

and the weight  $w$  of the semi-invariant which is the leading coefficient  $C_0$  in the linear covariant, are connected by the relation  $i_1 p_1 + i_2 p_2 - w = 2w$ .

Here  $p_1 = 2, p_2 = 3, w = 1$ , whence  $2i_1 + 3i_2 - 1 = 2w$ . More generally, if  $m$  be any positive integer and  $n$  any positive odd integer, we have, from the conditions of linear covariancy,  $2mi_1 + 3ni_2 - 1 = 2w$ . Thus the binary quadratic and cubic have an indefinite number of linear covariants.

\*Elliott's *Algebra of Quantics*. †*Ibid*.

We now shall find the linear covariants,

$$\begin{array}{ll} (a) & (2; 1, 2; 1, 3); \\ (c) & (5; 1, 2; 3, 3); \end{array} \quad \begin{array}{ll} (b) & (3; 2, 2; 1, 3); \\ (d) & (6; 2, 2; 3, 3). \end{array}$$

(a). Assume for the semi-invariant the most general form

$$S \equiv C_0 \equiv a_2 b_2 + \lambda_1 a_1 b_1 + \lambda_2 a_2 b_0,$$

where  $\lambda_1, \lambda_2$  are arbitrary multipliers. Operate on this with the two annihilators

$$\Omega_1 \equiv a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2}, \quad \Omega_2 \equiv b_0 \frac{d}{db_1} + 2b_1 \frac{d}{db_2} + 3b_2 \frac{d}{db_3},$$

and we obtain

$$a_0 b_1 [\lambda_1 + 2] + a_1 b_0 [2\lambda_2 + \lambda_1];$$

for which to vanish we must have

$$\lambda_1 + 2 = 0, \quad 2\lambda_2 + \lambda_1 = 0; \quad i. e., \quad \lambda_1 = -2\lambda_2 = -2.$$

$$\therefore C_0 \equiv a_0 b_2 - 2a_1 b_1 + a_2 b_0,$$

$$C_1 \equiv O_1 C_0 + O_2 C_0 \equiv a_0 b_3 - 2a_1 b_2 + a_2 b_1,$$

where we have used the annihilators

$$O_1 \equiv 2a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1}, \quad O_2 \equiv 3b_1 \frac{d}{db_0} + 2b_2 \frac{d}{db_1} + b_3 \frac{d}{db_2}.$$

Thus the linear covariant is

$$I. \quad (a_0 b_2 - 2a_1 b_1 + a_2 b_0)x + (a_0 b_3 - 2a_1 b_2 + a_2 b_1)y.$$

The second transvectant of the quadratic and cubic gives I.

(b). Including all possible terms, the semi-invariant is of the form

$$S \equiv C_0 \equiv a_0^2 b_3 + \lambda_1 a_0 a_1 b_2 + \lambda_2 a_0 a_2 b_1 + \lambda_3 a_1^2 b_1 + \lambda_4 a_1 b_2 b_0.$$

Operate on this with  $\Omega_1, \Omega_2$ . The vanishing of the expression requires four linear equations in the  $\lambda$ 's, from which we find

$$\lambda_1 = -3, \quad \lambda_2 = 1, \quad \lambda_3 = 2, \quad \lambda_4 = -1.$$

$$\therefore S \equiv C_0 \equiv a_0^2 b_3 - 3a_0 a_1 b_2 + a_0 a_2 b_1 + 2a_1^2 b_1 - a_1 a_2 b_0,$$

$$C_1 \equiv O_1 C_0 + O_2 C_0 \equiv -(a_2^2 b_0 - 3a_1 a_2 b_1 + a_0 a_2 b_2 + 2a_1^2 b_2 - a_0 a_1 b_3).$$

The linear covariant is

II.

$$O_0x + C_1y.$$

The first transvectant of I and the quadratic gives II. The third transvectant of the cubic and the square of the quadratic gives II.

(c). Assume the semi-invariant to be

$$S \equiv C_0 \equiv a_0b_0b_2b_3 + \lambda_1a_0b_1^2b_3 + \lambda_2a_0b_1b_2^2 + \lambda_3a_1b_0b_1b_3 \\ + \lambda_4a_1b_0b_2^2 + \lambda_5a_1b_1^2b_2 + \lambda_6a_2b_0^2b_3 + \lambda_7a_2b_0b_1b_2 + \lambda_8a_2b_1^2b_2.$$

Solving as in (a), we obtain eight linear equations in the  $\lambda$ 's, from which we find

$$\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = 2, \lambda_4 = -4, \lambda_5 = 2, \lambda_6 = -1, \lambda_7 = 3, \lambda_8 = -2.$$

Thus the linear covariant is

III.

$$(a_0b_0b_2b_3 - 2a_0b_1^2b_3 + a_0b_1b_2^2 + 2a_1b_0b_1b_3 - 4a_1b_0b_2^2 \\ + 2a_1b_1^2b_2 - a_2b_0^2b_3 + 3a_2b_0b_1b_2 - 2a_2b_1^2b_2)x \\ - (a_2b_0b_1b_3 - 2a_2b_0b_2^2 + a_2b_1^2b_2 + 2a_1b_0b_2b_3 - 4a_1b_1^2b_3 \\ + 2a_1b_1b_2^2 - a_0b_0b_3^2 + 3a_0b_1b_2b_3 - 2a_0b_2^2b_3)y.$$

The first transvectant of I and the *Hessian* of the cubic gives III. The second transvectant of the quadratic and the *cubicovariant* of the cubic gives III.

(d). Here we assume the semi-invariant to be

$$S \equiv C_0 \equiv a_0^2b_0b_3^2 + \lambda_1a_0^2b_2^2 + \lambda_2a_0^2b_1b_2b_3 + \lambda_3a_0a_1b_0b_2b_3 + \lambda_4a_0a_1b_1^2b_3 \\ + \lambda_5a_0a_1b_1b_2^2 + \lambda_6a_0a_2b_0b_1b_3 + \lambda_7a_0a_2b_0b_2^2 + \lambda_8a_0a_2b_1^2b_2 \\ + \lambda_9a_1^2b_0b_1b_3 + \lambda_{10}a_1^2b_0b_2^2 + \lambda_{11}a_1^2b_1^2b_2 + \lambda_{12}a_1a_2b_0^2b_3 \\ + \lambda_{13}a_1a_2b_0b_1b_2 + \lambda_{14}a_1a_2b_1^3 + \lambda_{15}a_2^2b_0b_2 + \lambda_{16}a_2^2b_0b_1^2,$$

which includes all possible terms. As in (a), we obtain

$$a_0^2b_1^2b_3(\lambda_4 + 2\lambda_1) + a_0a_2b_0^2b_3(\lambda_{12} + \lambda_6) + a_0a_2b_1^3(\lambda_{14} + 2\lambda_8) + a_1^2b_0^2b_3(2\lambda_{12} + \lambda_9) \\ + a_1^2b_1^3(2\lambda_{14} + 2\lambda_{11}) + a_2^2b_0^2b_1(4\lambda_{16} + 4\lambda_{15}) + a_0^2b_0b_2b_3(\lambda_3 + \lambda_1 + 6) + a_0^2b_1b_2^2 \\ (\lambda_5 + 6\lambda_2 + 3\lambda_1) + a_1a_2b_0^2b_2(4\lambda_{15} + \lambda_{12} + 3\lambda_{12}) + a_1a_2b_0b_1^2(4\lambda_{16} + 3\lambda_{14} + 2\lambda_{12}) + \\ a_0a_1b_0b_1b_3(2\lambda_9 + 2\lambda_6 + 2\lambda_4 + 2\lambda_3) + a_0a_1b_0b_2^2(2\lambda_{10} + 2\lambda_7 + \lambda_5 + 3\lambda_8) + a_0a_1b_1^2b_2 \\ (2\lambda_{11} + 2\lambda_8 + 4\lambda_5 + 3\lambda_4) + a_0a_2b_0b_1b_2(\lambda_{13} + 2\lambda_8 + 4\lambda_7 + 3\lambda_6) + a_1^2b_0b_1b_2(2\lambda_{13} + 2\lambda_{11} \\ + 4\lambda_{10} + 3\lambda_9).$$

Its vanishing gives (5; 2, 2; 3, 3) relations which have to be satisfied by the (6; 2, 2; 3, 3) multipliers. If then (6; 2, 2; 3, 3) exceeds (5; 2, 2; 3, 3) we

can satisfy them, the number of the multipliers still arbitrary being  $(6; 2, 2; 3, 3) - (5; 2, 2; 3, 3) \equiv 17 - 15 = 2$ .

*First*, the first transvectant (1), of the cubic and the square of I; and (2), of the *Hessian* of the cubic and II: or, *second*, the second transvectant (1), of the cubic and the square of I; (2), of the quadratic and the product of I by the *Hessian* of the cubic; and (3), of the *Hessian* of the cubic and the product of the quadratic by I: or, *third*, the third transvectant of the *cubicovariant* of the cubic and the square of the quadratic—either *first* or *second* or *third* shows that the partitions\*  $a_2^2 b_0 b_2$  and  $a_2^2 b_0 b_1^2$  are absent from this linear covariant. We then have  $\lambda_{16} = \lambda_{15} = 0$ . Now from the other fourteen linear equations in the  $\lambda$ 's we obtain

$$\begin{aligned} \lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -3, \quad \lambda_4 = 6, \quad \lambda_5 = -3, \quad \lambda_6 = -1, \quad \lambda_7 = 2, \\ \lambda_8 = -1, \quad \lambda_9 = -2, \quad \lambda_{10} = 4, \quad \lambda_{11} = -2, \quad \lambda_{12} = 1, \quad \lambda_{13} = -3, \quad \lambda_{14} = 2. \end{aligned}$$

The required linear covariant is

$$\begin{aligned} (a_0^2 b_0 b_3 - 3a_0^2 b_1 b_2 b_3 + 2a_0^2 b_2^3 - 3a_0 a_1 b_0 b_2 b_3 + 6a_0 a_1 b_1^2 b_3 - 3a_0 a_1 b_1 b_2^2 \\ - a_0 a_2 b_0 b_1 b_3 + a_0 a_2 b_0 b_2^2 - a_0 a_2 b_1^2 b_2 - 2a_1^2 b_0 b_1 b_3 + 4a_1^2 b_0 b_2^2 - 2a_1^2 b_1^2 b_2 \\ + a_1 a_2 b_0^2 b_3 - 3a_1 a_2 b_0 b_1 b_2 + 2a_1 a_2 b_1^3)x + (a_0 a_1 b_0 b_3^2 - 3a_0 a_1 b_1 b_2 b_3 + 2a_0 a_1 b_2^3 \\ - a_0 a_2 b_0 b_2 b_3 + 2a_0 a_2 b_1^2 b_2 - a_0 a_2 b_1 b_2^2 - 2a_1^2 b_0 b_2 b_3 + 4a_1^2 b_1^2 b_3 - 2a_1^2 b_1 b_2^2 \\ - 3a_1 a_2 b_0 b_1 b_3 + 6a_1 a_2 b_0 b_1^2 - 3a_1 a_2 b_1^2 b_2 + a_2^2 b_0^2 b_3 - 3a_2^2 b_0 b_1 b_2 + 2a_2^2 b_1^3)y. \end{aligned}$$

The number of linearly independent semi-invariants of given weight  $w$  and partial degrees  $i_1, i_2$  of the quadratic and cubic, is given by

$$(w; i_1, p_1; i_2, p_2) - (w-1; i_1, p_1; i_2, p_2).$$

This expression is not applicable when it exceeds unity for all values of  $m$  and  $n$  that do not give linear semi-invariants, because to each of these values corresponds one and only one linear covariant for a transvection of some combination (1), of the two quantities; or (2), of their covariants; or (3), of the quantities and their covariants. Professor Paul Gordan has proved that a complete system of transvectants is coextensive with a complete system of covariants, also including invariants as a particular case. For the geometrical interpretation of this system of quantities, see Art. 198 of Salmon's *Higher Algebra*.

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\* For the theory of numbers of partitions, see Professor Cayley's second memoir on Quantics (*Collected Works*, Vol. II).